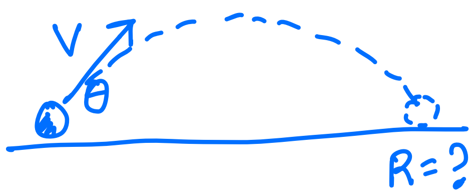
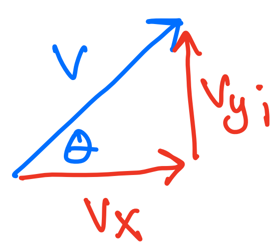
A projectile is fired across level ground with an initial speed of V at an initial angle of θ. Where does it land? (In other words, what is the range of the projectile?)

Our answer can only have the letters given in the problem - so "V" and "θ" are the only allowed variables. Since "g" is a constant, that is also allowed in our answer.



When doing projectile motion, we need the components of the initial velocity - so do that first:



$$V_x = V\cos\theta$$

$$V_{y_i} = V\sin\theta$$

All projectile motion is based on the following basic equations:

$$x = V_x t \qquad y = -\tfrac{1}{2}gt^2 + V_{y_i}t + y_i \qquad V_y = -gt + V_{y_i}$$

So for this problem we have

$$x = V\cos\theta\, t$$

$$y = -\tfrac{1}{2}gt^2 + V\sin\theta\, t \qquad (y_i = 0)$$

$$V_y = -gt + V\sin\theta$$

Now let's actually try & solve.

$X = v\cos\theta\, t \longrightarrow R = v\cos\theta\, T$ where "T" means the total time in the air

Notice "V" & "θ" are allowed in our answer, but not the "T". That means we need to find the total time in the air. Easy! There are 3 ways:

Option 1| $y = 0$ when it lands, so

$y = -\frac{1}{2}gt^2 + v\sin\theta t \rightarrow 0 = -\frac{1}{2}gT^2 + v\sin\theta T$

$T = \frac{2v\sin\theta}{g}$

option 2| $V_y = -V_{y_i}$ when it lands, so

$V_y = -gt + v\sin\theta \rightarrow -v\sin\theta = -gT + v\sin\theta$

$T = \frac{2v\sin\theta}{g}$

option 3| $V_y = 0$ @ Maximum Height, which happens at time $T/2$ in this case.

$V_y = -gt + v\sin\theta \rightarrow 0 = -g\left(\frac{T}{2}\right) + v\sin\theta$

$T = \frac{2v\sin\theta}{g}$

Almost Done!

So we have $R = (v\cos\theta)T$ & $T = \frac{2v\sin\theta}{g}$

So $R = \frac{v^2\, 2\sin\theta\cos\theta}{g}$

$\sin 2\theta = 2\sin\theta\cos\theta$

So $R = \frac{v^2 \sin 2\theta}{g}$

Done! BUT This equation is ONLY true when projectile is fired across level ground from the ground. It does not work if the inital & final heights are different!

ALSO remember that it is based off the initial velocity!

1. So for a projectile fired across level ground ($y_i = y_f = 0$) what initial angle would maximize the range?

$$R = \frac{v^2 \sin 2\theta}{g}$$

The part that depends on θ is "$\sin 2\theta$". The maximum value that can be is 1, so

$$1 = \sin 2\theta$$

So $2\theta = 90^\circ$

$\theta = 45^\circ$

If you want to be fancier, find θ to maximize R:

$$\frac{dR}{d\theta} = \frac{d}{d\theta}\left(\frac{v^2 \sin 2\theta}{g}\right) = \frac{v^2}{g}(-\cos 2\theta)(2)$$

$$\frac{dR}{d\theta} = -\frac{2v^2}{g}\cos(2\theta)$$

set derivative to 0 to find max & mins:

$$0 = -\frac{2v^2}{g}\cos(2\theta)$$

$$\cos(2\theta) = 0$$

$$2\theta = 90^\circ$$

$\theta = 45^\circ$

(First way is much easier)

② What is the relationship between 2 angles that have the same range?

Look at the equation we got along the derivation

$$R = \frac{v^2 2\sin\theta\cos\theta}{g}$$

So if $R_1 = R_2$

$$\cancel{\frac{v^2}{g}}\cancel{2}\sin\theta_1\cos\theta_1 = \cancel{\frac{v^2}{g}}\cancel{2}\sin\theta_2\cos\theta_2$$

$$\sin\theta_1\cos\theta_1 = \sin\theta_2\cos\theta_2$$

Hey! Remember that (for $0^\circ \leq \theta \leq 90^\circ$)

$$\sin(\theta) = \cos(90-\theta)$$

and $\cos(\theta) = \sin(90-\theta)$

So if $\theta_2 = 90 - \theta_1$, then

$$\sin\theta_2\cos\theta_2$$

$$= \sin(90-\theta_1)\cos(90-\theta_1)$$

$$= \cos\theta_1 \quad \sin\theta_1$$

So $\sin\theta_1\cos\theta_1 = \cos\theta_1\sin\theta$:)

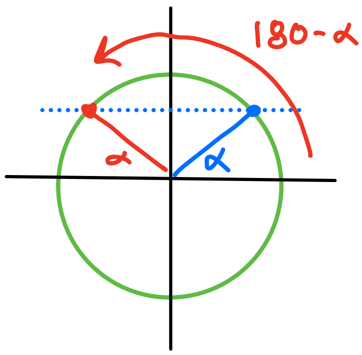
So θ_1 & θ_2 are complementary or you could be more formal:

$R_1 = R_2$ $\quad \frac{v^2}{g}\sin(2\theta_1) = \frac{v^2}{g}\sin(2\theta_2)$

$$\sin(2\theta_1) = \sin(2\theta_2)$$

On the unit circle, if $\sin\alpha = \sin\beta$

then $\beta = 180 - \alpha$



$$\therefore\ 2\theta_1 = 180 - 2\theta_2$$

$$\theta_1 = 90 - \theta_2$$

$$\theta_1 + \theta_2 = 90^\circ$$